# Non-Unitary and Unitary Transitions in Generalized Quantum Mechanics and Information Problem Solving

A.E.Shalyt-Margolin *

National Center of Particles and High Energy Physics, Bogdanovich Str. 153, Minsk 220040, Belarus

PACS: 03.65; 05.20
Keywords:deformed density matrices, deformed wave-function, deformed Heisenberg algebra,unitarity,information problem

**Abstract**

The present work is a study of the unitarity problem for Quantum Mechanics at Planck Scale considered as Quantum Mechanics with Fundamental Length (QMFL).In the process QMFL is described as deformation of a well-known Quantum Mechanics (QM). Similar to previous works of the author, the basic approach is based on deformation of the density matrix (density pro-matrix) with concurrent development of the wave function deformation in the respective Schrodinger picture. It is demonstrated that the existence of black holes in the suggested approach in the end twice results in nonunitary transitions (first after the Big Bang of QMFL to QM, and then when on trapping of the matter into the black hole the situation is just the opposite - from QM to QMFL)and hence in recovery of the unitarity. In parallel this problem is considered in the deformation terms of Heisenberg algebra, showing the identity of the basic results. From this an explicit solution for Hawking's informaion paradox has been derived.

*Fax: (+375) 172 326075; e-mail: a.shalyt@mail.ru;alexm@hep.by

# 1 Introduction

As is known, the Early Universe Quantum Mechanics (Quantum Mechanics at Planck scale) is distinguished from a well- known Quantum Mechanics at conventional scales [1],[2]by the fact that in the first one the Generalized Uncertainty Relations (GUR)are fulfilled resulting in the emergence of a fundamental length, whereas in the second one the usual Heisenberg Uncertainty Relations are the case. In case of Quantum Mechanics with Fundamental Length (QMFL) all three well-known fundamental constants are present $G$,$c$ $\hbar$, while the classical QM is associated only with a single one $\hbar$. It is obvious that transition from the first to the second one within the inflation expansion is a nonunitary process, i.e. the process where the probabilities are not retained [3], [4].

Because of this, QMFL is considered as a deformation of QM. The deformation in Quantum Mechanics at Planck scale takes different paths: commutator deformation or more precisely deformation of the respective Heisenberg algebra [5],[6],[7] , i.e. the density matrix deformation approach,developed by the author with co-workers in a number of papers [3],[4],[8],[9],[10]. The first approach suffers from two serious disadvantages: (1)the deformation parameter is a dimensional variable $\kappa$ with a dimension of mass [5]; (2)in the limiting transition to QM this parameter goes to infinity and fluctuations of other values are hardly sensitive to it. Being devoid of the above limitation, the second approach by the author's opinion is intrinsic for QMFL: with it in QMFL the deformation parameter is represented by the dimensionless quantity $\alpha = l_{min}^2/x^2$, where $x$ is the scale and the variation interval $\alpha$ is finite $0 < \alpha \leq 1/4$ [3], [4],[10]. Besides, this approach contributes to the solution of particular problems such as the information paradox problem of black holes [3] and also the problem of an extra term in Loiuwille equation [8],[9],[10], derivation of Bekenstein-Hawking formula from the first principles [10], hypothesis of cosmic censorship [9],[10], more exact definition and expansion of the entropy notion through the introduction of the entropy density per unit minimum area [9],[10],[11]. Moreover, it is demonstrated that there exists a complete analogy in the construction and properties of quantum mechanics and statistical density matrices at Planck scale (density pro-matrices). It should be noted that an ordinary statistical density matrix appears in the low-temperature limit (at temperatures much lower than the Planck's)[12].In the present work the unitarity problem for QMFL is considered on the ba-

sis of this approach. It is shown that as distinct from Hawking's approach, in this treatment the existence of black holes is not the reason for the unitarity violation, rather being responsible for its recovery. First after the Big Bang (initial singularity)expansion of the Universe is associated with the occurrence of a nonunitary transition from QMFL to QM, and with trapping of the matter by the black hole (black hole singularity)we have a reverse nonunitary process from QM to QMFL. In such a manner a complete transition process from QMFL to the unitarity may be recovered. Thus, the existence of black holes contributes to the reconstruction of a symmetry of the general picture. Similar results may be obtained in terms of the Heisenberg's algebra deformation. So the problem of Hawking information paradox is solved by the proposed approach: the information quantity in the Universe is preserved. This paper is a summing-up of the tentative results obtained by the author on the information paradox as an extension of the earlier works [3] [11].

# 2 Some Preliminary Facts

In this section the principal features of QMFL construction are briefly outlined first in terms of the density matrix deformation (von Neumann's picture) and subsequently in terms of the wave function deformation (Schrodinger picture) [3],[4],[9],[10]. As mentioned above, for the fundamental deformation parameter we use $\alpha = l_{min}^2/x^2$, where $x$ is the scale.

**Definition 1. (Quantum Mechanics with Fundamental Length [for Neumann's picture])**

Any system in QMFL is described by a density pro-matrix of the form

$$\rho(\alpha) = \sum_{\mathbf{i}} \omega_{\mathbf{i}}(\alpha)|\mathbf{i}><\mathbf{i}|,$$

where

1. $0 < \alpha \leq 1/4$;
2. The vectors $|i>$ form a full orthonormal system;
3. $\omega_i(\alpha) \geq 0$ and for all $i$ the finite limit $\lim\limits_{\alpha \rightarrow 0} \omega_i(\alpha) = \omega_i$ exists;

4. $Sp[\rho(\alpha)] = \sum_i \omega_i(\alpha) < 1$, $\sum_i \omega_i = 1$;

5. For every operator $B$ and any $\alpha$ there is a mean operator $B$ depending on $\alpha$:

$$< B >_\alpha = \sum_i \omega_i(\alpha) < i|B|i > .$$

Finally, the following condition must be fulfilled:

$$Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] \approx \alpha. \tag{1}$$

Consequently we can find the value for $Sp[\rho(\alpha)]$ satisfying the condition of definition 1:

$$Sp[\rho(\alpha)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}. \tag{2}$$

According to point 5), $< 1 >_\alpha = Sp[\rho(\alpha)]$. Therefore for any scalar quantity $f$ we have $< f >_\alpha = fSp[\rho(\alpha)]$. We denote the limit $\lim\limits_{\alpha\to 0} \rho(\alpha) = \rho$ as the density matrix. Evidently, in the limit $\alpha \to 0$ we return to QM.

As was shown in [3],[9],[10]:

I. The above limit covers both Quantum and Classical Mechanics.

II. Density pro-matrix $\rho(\alpha)$ tests singularities. As a matter of fact, the deformation parameter $\alpha$ should assume the value $0 < \alpha \leq 1$. However, as seen from (2), $Sp[\rho(\alpha)]$ is well defined only for $0 < \alpha \leq 1/4$, i.e. for $x = il_{min}$ and $i \geq 2$ we have no problems at all. At the point, where $x = l_{min}$ (that corresponds to a singularity of space), $Sp[\rho(\alpha)]$ takes the complex values.

III. It is possible to read the equation (1) as

$$Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] = \alpha + a_0\alpha^2 + a_1\alpha^3 + ... \tag{3}$$

Then for example, one of the solutions of (1) is

$$\rho^*(\alpha) = \sum_i \alpha_i exp(-\alpha)|i >< i|, \tag{4}$$

, where all $\alpha_i > 0$ are independent of $\alpha$ and their sum is equal to 1 . In this way $Sp[\rho^*(\alpha)] = exp(-\alpha)$. Note that in the momentum representation $\alpha = p^2/p_{max}^2$, $p_{max} \sim p_{pl}$,where $p_{pl}$ is the Planck

momentum. When present in matrix elements, $exp(-\alpha)$ can damp the contribution of great momenta in a perturbation theory. The solution (1)given by the formula (4) is further referred to as **(exponential ansatz)**.This ansatz will be the principal one in our further consideration.

In[9],[10] it has been demonstrated, how a transition from Neumann's picture to Shrödinger's picture, i.e. from the density matrix deformation to the wave function deformation, may be realized by the proposed approach
**Definition 2. (Quantum Mechanics with Fundamental Length [Shrödinger's picture])**

Here, the prototype of Quantum Mechanical normed wave function (or the pure state prototype) $\psi(q)$ with $\int |\psi(q)|^2 dq = 1$ in QMFL is $\theta(\alpha)\psi(q)$. The parameter of deformation $\alpha$ assumes the value $0 < \alpha \leq 1/4$. Its properties are $|\theta(\alpha)|^2 < 1, \lim\limits_{\alpha \to 0} |\theta(\alpha)|^2 = 1$ and the relation $|\theta(\alpha)|^2 - |\theta(\alpha)|^4 \approx \alpha$ takes place. In such a way the total probability always is less than 1: $p(\alpha) = |\theta(\alpha)|^2 = \int |\theta(\alpha)|^2 |\psi(q)|^2 dq < 1$ and it tends to 1 when $\alpha \to 0$. In the most general case of the arbitrarily normed state in QMFL(mixed state prototype) $\psi = \psi(\alpha, q) = \sum_n a_n \theta_n(\alpha)\psi_n(q)$ with $\sum_n |a_n|^2 = 1$ the total probability is $p(\alpha) = \sum_n |a_n|^2 |\theta_n(\alpha)|^2 < 1$ and $\lim\limits_{\alpha \to 0} p(\alpha) = 1$.

It is natural that Shrodinger equation is also deformed in QMFL. It is replaced by the equation

$$\frac{\partial \psi(\alpha, q)}{\partial t} = \frac{\partial [\theta(\alpha)\psi(q)]}{\partial t} = \frac{\partial \theta(\alpha)}{\partial t}\psi(q) + \theta(\alpha)\frac{\partial \psi(q)}{\partial t}, \tag{5}$$

, where the second term in the right side generates the Shrodinger equation since

$$\theta(\alpha)\frac{\partial \psi(q)}{\partial t} = \frac{-i\theta(\alpha)}{\hbar} H\psi(q). \tag{6}$$

Here $H$ is the Hamiltonian and the first member is added, similarly to the member that appears in the deformed Loiuville equation, and vanishes when $\theta[\alpha(t)] \approx const$. In particular, this takes place in the low energy limit in QM, when $\alpha \to 0$. It should be noted that the above theory is not a time reversal of QM as the combination $\theta(\alpha)\psi(q)$ breaks down this property in the deformed Shrodinger equation. Time-reversal is conserved only in the low energy limit, when a quantum mechanical Shrodinger equation is valid.

## 3 Some Comments and Unitarity in QMFL

As has been indicated in the previous section, time reversal is retained in the large-scale limit only. The same is true for the superposition principle in Quantum Mechanics: indeed it may be retained in a very narrow interval of cases for the functions $\psi_1(\alpha, q) = \theta(\alpha)\psi_1(q) \quad \psi_2(\alpha, q) = \theta(\alpha)\psi_2(q)$ with the same value $\theta(\alpha)$. However, as for all $\theta(\alpha)$, their limit is $\lim\limits_{\alpha\to 0} |\theta(\alpha)|^2 = 1$ or equivalently $\lim\limits_{\alpha\to 0} |\theta(\alpha)| = 1$, in going to the low-energy limit each wave function $\psi(q)$ is simply multiplied by the phase factor $\theta(0)$. As a result we have Hilbert Space wave functions in QM. Comparison of both pictures (Neumann's and Shrodinger's), is indicative of the fact that the unitarity means the retention of the probabilities $\omega_i(\alpha)$ or retention of the squared modulus (and hence the modulus) for the function $\theta(\alpha)$: $|\theta(\alpha)|^2$,$(|\theta(\alpha)|)$.That is

$$\frac{d\omega_i[\alpha(t)]}{dt} = 0$$

or

$$\frac{d|\theta[\alpha(t)]|}{dt} = 0.$$

In this way a set of unitary transformations of QMFL includes a group $U$ of the unitary transformations for the wave functions $\psi(q)$ in QM.
It is seen that on going from Planck's scale to the conventional one , i.e. on transition from the early Universe to the current one, the scale has been rapidly changing in the process of inflation expansion and the above conditions failed to be fulfilled:

$$\frac{d\omega_i[\alpha(t)]}{dt} \neq 0, \frac{d|\theta[\alpha(t)]|}{dt} \neq 0. \tag{7}$$

In terms of the density pro-matrices of section 2 this is a limiting transition from the density pro-matrix in QMFL $\rho(\alpha)$,$\alpha > 0$ , that is a prototype of the pure state at $\alpha \to 0$, to the density matrix $\rho(0) = \rho$ representing a pure state in QM. Mathematically this means that a nontotal probability (below 1) is changed by the total one (equal to 1). For the wave functions in Schrodinger picture this limiting transition from QMFL QM is as follows:

$$\lim_{\alpha\to 0}\theta(\alpha)\psi(q)=\psi(q)$$

up to the phase factor

It is apparent that the above transition from QMFL to QM is not a unitary process, as indicated in [3],[4],[8]-[10]. However, the unitarity may be recovered when we consider in a sense a reverse process: absorption of the matter by a black hole and its transition to singularity that conforms to the reverse and nonunitary transition from QM to QMFL. Thus, nonunitary transitions occur in this picture twice:

$$I.(QMFL, OS, \alpha\approx 1/4) \stackrel{Big\ \ Bang}{\longrightarrow} (QM, \alpha\approx 0)$$

$$II.(QM, \alpha\approx 0) \stackrel{absorbing\ \ BH}{\longrightarrow} (QMFL, SBH, \alpha\approx 1/4),$$

Here the following abbreviations are used: OS for the Origin Singularity; BH for a Black Hole ; SBH for the Singularity in Black Hole.
As a result of these two nonunitary transitions the total unitarity may be recovered:

$$III.(QMFL, OS, \alpha\approx 1/4)\longrightarrow(QMFL, SBH, \alpha\approx 1/4)$$

In such a manner the total information quantity in the Universe remains unchanged, i.e. no information loss occurs.
In terms of the deformed Liouville equation [8]-[10] we arrive to the expression with the same right-hand parts for $t_{initial}\sim t_{Planck}$ and $t_{final}$ (for

$\alpha \approx 1/4$).

$$\frac{d\rho[\alpha(t),t]}{dt} = \sum_i \frac{d\omega_i[\alpha(t)]}{dt}|i(t)><i(t)| -$$
$$-i[H,\rho(\alpha)] = d[ln\omega(\alpha)]\rho(\alpha) - i[H,\rho(\alpha)]. \quad (8)$$

It should be noted that for the closed Universe one can consider Final Singularity (FS) rather than the Singularity of Black Hole (SBH), and then the right-hand parts of the diagrams II III will be changed:

$$IIa.(QM,\alpha \approx 0) \stackrel{Big\ \ Crunch}{\longrightarrow} (QMFL,FS,\alpha \approx 1/4),$$

$$IIIa.(QMFL,OS,\alpha \approx 1/4) \longrightarrow (QMFL,FS,\alpha \approx 1/4)$$

At the same time, in this case the general unitarity and information are still retained, i.e. we again have the unitary product of two nonunitary arrows:

$$IV.(QMFL,OS,\alpha \approx 1/4) \stackrel{Big\ \ Bang}{\longrightarrow} (QM,\alpha \approx 0) \stackrel{Big\ \ Crunch}{\longrightarrow} (QMFL,FS,\alpha \approx 1/4),$$

Finally, arrow III may appear immediately, i.e. without the appearance of arrows I and II, when in the Early Universe mini BH are arising:

$$IIIb.(QMFL,OS,\alpha \approx 1/4) \longrightarrow (QMFL,mini\ \ BH,SBH,\alpha \approx 1/4)$$

However, here, unlike the previous cases, the unitary transition occurs immediately, without any additional nonunitary ones and with retention of the total information.
And in terms of the entropy density matrix introduced in [11],

$$S^{\alpha_2}_{\alpha_1} = -Sp[\rho(\alpha_2)\ln(\rho(\alpha_1))] = -<\ln(\rho(\alpha_1))>_{\alpha_2},$$

retention of the information means that for any observer in the proper measurement scale $x_2$(with the deformation parameter $\alpha_2$)the densities of entropy in the vicinity of the initial and final singularity ($\alpha_1 \approx 1/4$) are coincident:

$$S(in) = S(out) = S^{\alpha_2}_{1/4}.$$

# 4 Unitarity, Non-Unitarity and Heisenbergs Algebra Deformation

The above-mentioned unitary and nonunitary transitions may be described in terms of Heisenbergs algebra deformation (deformation of commutators) as well. We use the principal results and designations from [5].In the process the following assumptions are resultant: 1)The three-dimensional rotation group is not deformed; the angular momentum $\mathbf{J}$ satisfies the undeformed $SU(2)$ commutation relations, and the coordinate and momenta satisfy the undeformed commutation relations $[J_i, x_j] = i\epsilon_{ijk}x_k, [J_i, p_j] = i\epsilon_{ijk}p_k$. 2) The momenta commute between themselves: $[p_i, p_j] = 0$, so the translation group is also not deformed. 3) The $[x, x]$ and $[x, p]$ commutators depend on the deformation parameter $\kappa$ with dimensions of mass. In the limit $\kappa \to \infty$ when $\kappa$ is much larger than any energy, the canonical commutation relations are recovered.
For a specific realization of points 1) to 3) the generating GUR are of the form [5]: ($\kappa$-deformed Heisenberg algebra)

$$[x_i, x_j] = -\frac{\hbar^2}{\kappa^2}\, i\epsilon_{ijk}J_k \qquad (9)$$

$$[x_i, p_j] = i\hbar\delta_{ij}(1 + \frac{E^2}{\kappa^2})^{1/2}\,. \qquad (10)$$

Here $E^2 = p^2 + m^2$. Note that in this formalism the transition from GUR to UR, or equally from QMFL to QM with $\kappa \to \infty$ ,from Planck scale to the conventional one, is nonunitary, exactly following the transition from

density pro-matrix to the density matrix in previous sections:

$$\rho(\beta \neq 0) \stackrel{\beta \rightarrow 0}{\longrightarrow} \rho$$

Then the first arrow I in the formalism of this section may be as follows:

$$I'.(GUR, OS, \kappa \sim M_p) \stackrel{Big\_Bang}{\longrightarrow} (UR, \kappa = \infty)$$

or what is the same

$$I''.(QMFL, OS, \kappa \sim M_p) \stackrel{Big\_Bang}{\longrightarrow} (QM, \kappa = \infty),$$

where $M_p$ is the Planck mass. In some works of the last two years Quantum Mechanics for a black hole has been already considered as a Quantum Mechanics with GUR [13]-[15]. As a consequence, by this approach the black hole is not completely evaporated but rather some stable remnants always remain in the process of its evaporation with a mass $\sim M_p$. In terms of [5] this means nothing else but a reverse transition: $(\kappa = \infty) \rightarrow (\kappa \sim M_p)$. And for an outside observer this transition is of the form:

$$II'.(UR, \kappa = \infty) \stackrel{absorbing\ BH}{\longrightarrow} (GUR, SBH, \kappa \sim M_p),$$

$$II''.(QM, \kappa = \infty) \stackrel{absorbing\ BH}{\longrightarrow} (QMFL, SBH, \kappa \sim M_p),$$

So similar to the previous section, two nonunitary mutually reverse transitions: a)$I', (I'')$;b)$II', (II'')$ are liable to generate a unitary transition:

$$III'.(GUR, OS, \kappa \sim M_p) \stackrel{Big\_Bang}{\longrightarrow} (UR, \kappa = \infty) \stackrel{absorbing\ BH}{\longrightarrow} (GUR, SBH, \kappa \sim M_p)$$

or to summerize

$$III''.(GUR, OS, \kappa \sim M_p) \rightarrow (GUR, SBH, \kappa \sim M_p)$$

In conclusion of this section it should be noted that an approach to the

Quantum Mechanics at Planck Scale using the Heisenberg algebra deformation (similar to the approach based on the density matrix deformation from the previous section) gives a deeper insight into the possibility of retaining the unitarity and the total quantity of information in the Universe, making possible the solution of Hawkings information paradox [16]-[18].

# 5 Conclusion

Thus, this work outlines that the existence of GUR and hence the appearance of QMFL not only enable a better understanding of the information problem in the Universe but also provides a key to the solution of this problem in a not inconsistent manner practically in the same way but irrespective of the approach: density matrix deformation or Heisenberg algebra deformation.
It should be noted that the question of the relationship between these two approaches, i.e. transition from one deformation to the other, still remains open. This aspect is to be studied in further investigations of the author.

# References


[1] D. V. Ahluwalia,Quantum Measurement, Gravitation, and Locality, Phys.Lett. B339 (1994)301

[2] D.V.Ahluwalia,Wave-Particle duality at the Planck scale: Freezing of neutrino oscillations, Phys.Lett. A275 (2000)31;Interface of Gravitational and Quantum Realms, Mod.Phys.Lett. A17(2002)1135

[3] A.E.Shalyt-Margolin, Fundamental Length,Deformed Density Matrix and New View on the Black Hole Information Paradox,[gr-qc/0207074]

[4] A.E.Shalyt-Margolin and A.Ya.Tregubovich, Generalized Uncertainty Relations,Fundamental Length and Density Matrix,[gr-qc/0207068]

[5] M.Maggiore, The algebraic structure of the generalized uncertainty principle,Phys.Lett.B319(1993)83

[6] M.Maggiore,Quantum Groups,Gravity and Generalized Uncertainty Principle, Phys.Rev.D49(1994)5182

[7] S.Capozziello,G.Lambiase and G.Scarpetta, The Generalized Uncertainty Principle from Quantum Geometry, Int.J.Theor.Phys. 39 (2000),15

[8] A.E.Shalyt-Margolin and J.G.Suarez. Density Matrix and Dynamical aspects of Quantum Mechanics with Fundamental Length, [gr-qc/0211083]

[9] A.E.Shalyt-Margolin and J.G.Suarez,Quantum Mechanics of the Early Universe and its Limiting Transition,[gr-qc/0302119]

[10] A.E.Shalyt-Margolin and J.G.Suarez,Quantum Mechanics at Planck's scale and Density Matrix,Intern.Journ.of Mod.Phys.D.12(2003)1265

[11] A.E.Shalyt-Margolin. Deformed density matrix, Density of entropy and Information problem,[gr-qc/0307096].

[12] A.E.Shalyt-Margolin,Density Matrix in Quantum and Statistical Mechanics at Planck-Scale [gr-qc/0307056]

[13] R.Adler,P.Chen and D.Santiago,The Generalised Uncertainty Principle and Black Hole Remnants,Gen.Rel.Grav.33(2001)2101

[14] P.Chen and R.Adler, Black Hole Remnants and Dark Matter,Nucl.Phys.Proc.Suppl.124(2003)103

[15] P.S.Custodio,J.E.Horvath,The Generalized Uncertainty Principle,entropy bounds and black hole (non-)evaporation in thermal bath,Class.Quant.Grav.20(2003)L197

[16] S.Hawking,Breakdown of Predictability in Gravitational Collapse, Phys.Rev.D14(1976)2460

[17] S.Giddings,The Black Hole Information Paradox,[hep-th/9508151]

[18] A.Strominger, Les Houches Lectures on Black Holes, [hep-th/9501071]